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1989 J. Phys. A: Math. Gen. 22 L419

(http://iopscience.iop.org/0305-4470/22/10/001)

J. Phys. A: Math. Gen. 22 (1989) L419–L423. Printed in the UK

LETTER TO THE EDITOR

Bound states and confining properties of relativistic point interaction potentials

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Received 6 February 1989

Abstract. Bound states of the one-dimensional Dirac equation for vector plus Lorentz scalar point interaction potentials have been obtained. Confining properties of these potentials are briefly discussed.

Point interaction potentials (PIPS) may be used to approximate, in a simple way, more structured and more complex, short-ranged potentials. Calculations involving PIPS, usually represented as δ-function potentials, are largely simplified. Although solutions of the Schrödinger equation for δ-function potentials are quite straightforward, some ambiguities have been found in defining relativistic PIPS (Fairbairn *et al* 1973, Sutherland and Mattis 1981). Potentials of different shapes which approach the δ-function limit (zero width and constant area) give eigenfunctions reaching different values at the discontinuity point. A reasonable criterion to surmount this ambiguity was given recently by McKellar and Stephenson (1987a, b). These authors considered the solution of the one-dimensional Dirac equation for an electrostatic (vector type) plus a Lorentz scalar square barrier, and then they studied the δ-function limit. Also, they obtained boundary conditions for the wavefunctions of relativistic particles in pure vector and pure scalar sharply peaked potentials, which approach the δ-function limit. The results are independent of how this limit is taken in the potential after solving the Dirac equation. Moreover, both types of potentials have been separately considered in their discussions on quark confinement by a periodic array of δ-function potentials (a Dirac–Kronig–Penney model for relativistic quarks in nuclei).

One of the aims of the present letter is to extend previous results to be applied for any peaked mixed potential. Boundary conditions and confining properties for the one-dimensional Dirac equation with mixed PIPS are briefly discussed, especially when the strength of vector and scalar terms of the potential just take the same value; this choice of the potential parameters could be of particular interest in quarkonium physics (Beavis *et al* 1979).

Since the advent of quark models, interest in bound states of relativistic systems has increased. The relativistic covariance of the instantaneous two-body Dirac equation (a simplified formulation of the relativistic two-body problem) restricts the interaction to the form of a PIP (Glöckle *et al* 1987, and references therein), because only a mathematical point has a relativistically invariant shape. As a first approximation, the

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0305-4470/89/100419+05$02.50

one-body Dirac equation may be used to find bound states when the mass of one of the constituent particles becomes very large (for instance, mesons with one of the quarks heavier than the other do exist). We also discuss bound states of the Dirac equation for an arbitrary mixture of vector and scalar PIPs.

Let us start with the one-dimensional Dirac equation

$$\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}=\hat{K}(x)\psi(x) \tag{1}$$

where the operator $\hat{K}(x)=-\mathrm{i}\sigma_x(\sigma_z m+U(x)-E)$ acts on the two-component wavefunction of the particle, the σ being 2×2 Pauli matrices. We take a mixed potential of the form

$$U(x)=(g_\mathrm{v}+\sigma_z g_\mathrm{s})v(x) \tag{2}$$

where $v(x)$ is any peaked function at $x=0$ satisfying $\int_{-\infty}^{\infty} v(x)\,\mathrm{d}x=1$. g_v and g_s are the strengths of the vector and scalar components of the potential, respectively. The first-order differential equation (1) can be solved by an iterative Neumann solution. Around the PIP localisation, the solution becomes

$$\psi(0^+)=P\exp\left(\int_{0^-}^{0^+}\hat{K}(x)\,\mathrm{d}x\right)\psi(0^-) \tag{3}$$

P being a Dyson-type ordering operator. Taking the δ-function limit of $v(x)$, the integration equals $-\mathrm{i}\sigma_x(g_\mathrm{v}+\sigma_z g_\mathrm{s})$, which commutes at separate spatial points so we may set $P=1$. Finally, the following boundary condition is reached:

$$\psi(0^+)=\exp[-\mathrm{i}\sigma_x(g_\mathrm{v}+\sigma_z g_\mathrm{s})]\psi(0^-). \tag{4}$$

In order to obtain an explicit expression of boundary condition, we use the Lagrange-Sylvester formula

$$f(\mathbf{M})=f(\lambda_1)\frac{\lambda_2-\mathbf{M}}{\lambda_2-\lambda_1}+f(\lambda_2)\frac{\lambda_1-\mathbf{M}}{\lambda_1-\lambda_2} \tag{5}$$

where λ_1 and λ_2 are the eigenvalues of the 2×2 matrix $\mathbf{M}$. Thus, taking $\mathbf{M}=-\mathrm{i}\sigma_x(g_\mathrm{v}+\sigma_z g_\mathrm{s})$, whose eigenvalues are $\pm\mathrm{i}(g_\mathrm{v}^2-g_\mathrm{s}^2)^{1/2}$, we obtain

$$\psi(0^+)=\cos(g_\mathrm{v}^2-g_\mathrm{s}^2)^{1/2}\begin{pmatrix}1 & -\mathrm{i}\alpha_-\\ -\mathrm{i}\alpha_+ & 1\end{pmatrix}\psi(0^-) \tag{6}$$

where $\alpha_\pm=(g_\mathrm{v}\pm g_\mathrm{s})\tan(g_\mathrm{v}^2-g_\mathrm{s}^2)^{1/2}/(g_\mathrm{v}^2-g_\mathrm{s}^2)^{1/2}$ are always real numbers. Note that the current $\psi^\dagger\sigma_x\psi$ is continuous at $x=0$, but the probability $\psi^\dagger\psi$ jumps except for pure vector potentials. Equation (6) cannot be applied for equally mixed potentials ($g_\mathrm{v}=g_\mathrm{s}$), since eigenvalues of $\mathbf{M}=-\mathrm{i}g_\mathrm{v}\sigma_x(1+\sigma_z)$ vanish. Fortunately, the condition $\mathbf{M}^n=0$ is fulfilled for $n\geqslant 2$, for this particular mixture of the potentials, so the exponential function in equation (5) is easily expanded to give $\exp(\mathbf{M})=1-\mathrm{i}g_\mathrm{v}\sigma_x(1+\sigma_z)$. This result could be regarded as a limiting case of equation (6), considering that α_+ approaches $2g_\mathrm{v}$ and α_- vanishes as $g_\mathrm{v}\to g_\mathrm{s}$. Analogous arguments hold for inverted mixed potentials ($g_\mathrm{v}=-g_\mathrm{s}$). Therefore, equation (6) may be used even if $|g_\mathrm{v}|=|g_\mathrm{s}|$. We should emphasise that periodic trigonometric functions appearing in equation (6) switches in the form of hyperbolic functions when the scalar term is larger than the vector one, presenting no periodicity with the potential strength.

We shall discuss in detail the following special cases.

(a) Pure vector potential ($g_v \neq 0$, $g_s = 0$): $\alpha_+ = \alpha_- = \tan g_v$.

(b) Pure scalar potential ($g_v = 0$, $g_s \neq 0$): $\alpha_+ = -\alpha_- = \tanh g_s$.

(c) Equally mixed potential ($g_v = g_s \equiv g_+/2$): $\alpha_+ = g_+$, $\alpha_- = 0$. Thus the upper component of the wavefunction becomes continuous, like the non-relativistic wavefunction for the δ-function potential. Nevertheless, the lower component remains discontinuous.

(d) Inverted mixed potential ($g_v = -g_s \equiv g_-/2$): $\alpha_+ = 0$, $\alpha_- = g_-$. In this case, the lower component is continuous at the localisation of the PIP.

In order to find bound states of the one-dimensional Dirac equation for the potential $U(x)$ given in (2), we employ the S-matrix formalism. A free particle coming from the left may be reflected at the barrier, and also transmission may exist. For a normalised-to-unity incident particle wavefunction, solution of the Dirac equation for $x \neq 0$ is written as

$$\psi(x) = \begin{cases} \mathbf{W}(p)\,e^{ipx} + R\mathbf{W}(-p)\,e^{-ipx} & x<0 \\ T\mathbf{W}(p)\,e^{ipx} & x>0 \end{cases} \tag{7}$$

where R and T are the amplitudes of the reflected and transmitted waves respectively, satisfying $|R|^2 + |T|^2 = 1$. The two-component spinor is defined as $\mathbf{W}^\dagger(p) = (1, p/(E+m))$ ($\dagger$ denotes the Hermitian conjugate). From the boundary condition (6) we find

$$T = \left\{\cos^2(g_v^2 - g_s^2)^{1/2}\left[1 + \frac{i}{2}\left(\frac{E+m}{p}\alpha_+ + \frac{p}{E+m}\alpha_-\right)\right]\right\}^{-1}. \tag{8}$$

The poles of T in the upper half p plane lie along the imaginary axis for potentials vanishing beyond some finite distance, such as PIPS. These poles correspond to the bound states of the potential. Therefore, replacing p by iq, where $q = +(m^2 - E^2)^{1/2}$ is real for bound states, we obtain

$$-q = (Eg_v + mg_s)\tan(g_v^2 - g_s^2)^{1/2}/(g_v^2 - g_s^2)^{1/2}. \tag{9}$$

It can be readily checked that this result does not depend on the particular choice of matrices appearing in the Dirac equation. Note that massless particles cannot be bounded, even if the potential strength becomes infinite. Using the normalisation condition $\int_{-\infty}^{\infty} \psi^\dagger(x)\psi(x)\,dx = 1$, bound state eigenfunctions can be written as

$$\psi(x) = [q(1+E/m)/(1+1/\beta^2)]^{1/2} \times \begin{cases} (1/\beta)\mathbf{W}(-iq)\,e^{qx} & x<0 \\ \mathbf{W}(iq)\,e^{-qx} & x>0 \end{cases} \tag{10}$$

with

$$\beta \equiv -\tfrac{1}{2}\cos(g_v^2 - g_s^2)^{1/2}\left(\frac{E+m}{q}\alpha_+ + \frac{q}{E+m}\alpha_-\right).$$

Let us now consider some particular cases listed below.

(a) Pure vector potentials ($g_s = 0$). The condition (9) becomes $-q = E\tan g_v$. Therefore, the particle energy becomes positive for negative values of $\tan g_v$, while a negative bound-state level appears for positive $\tan g_v$ (the special case $\tan g_v = 0$ will be discussed below). The sign of the strength g_v, positive for repulsive potentials and negative for attractive ones, is immaterial as far as the existence of bound states is concerned, in accordance with general results for bound states of the one-dimensional Dirac equation (Coutinho and Nogami 1987). The energy of the single bound state is

$$E = -\tfrac{1}{2}m\sin 2g_v/|\sin g_v|. \tag{11}$$

We observe the occurrence of discontinuity points at $g_v = n\pi$, n being an integer (this situation was first discussed by Sutherland and Mattis (1981)). Figure 1 shows the bound-state energy as a function of the potential strength. A bound state emerges from the continuum of positive energy states as g_v is slightly negative. By decreasing g_v, this bound state passes through zero and reaches the negative continuum at $g_v = -\pi$, just as another bound state drops out of the positive continuum. Analogous behaviour is found for positive values of g_v, but beginning from the continuum of negative energy. At the critical values of the potential strength $g_v = n\pi$, the boundary condition (6) is simply written $\psi(0^+) = (-1)^n \psi(0^-)$; except for the constant phase factor, the vector potential does not act on the wavefunction and there is no binding of particles.

(*b*) Pure scalar potentials. Now equation (9) is written as $-q = m \tanh g_s$. Since q must be kept positive, bound states occur only if $g_s < 0$; there exists no binding for repulsive scalar potentials at all. Pairs of allowed energy values appear, which is a common feature of other scalar-type potentials (Coutinho *et al* 1988). Bound states are given by

$$E = \pm m \operatorname{sech} g_s. \tag{12}$$

Unlike the vector coupling, the same scalar potential can bind particles and antiparticles alike. As seen in figure 1, the energy level never reaches zero (a comprehensive discussion on the zero-energy solution for general scalar potentials can be found in Coutinho *et al* (1988)). Therefore, positive and negative energy states remain well separated even if the potential becomes strong.

(*c*) Equally mixed potentials. Since $\alpha_+ = g_+$ and $\alpha_- = 0$ we obtain $-q = \frac{1}{2}(m + E)g_+$, so the condition for the existence of bound states is $g_+ < 0$. The corresponding energy level is given by

$$E = m(4 - g_+^2)/(4 + g_+^2) \tag{13}$$

and it becomes negative for $|g_+| > 2$. The boundary of the lower continuum is never reached for finite values of g_+ due to the presence of the scalar potential term, but the energy level crosses zero because of the vector potential term (figure 1).

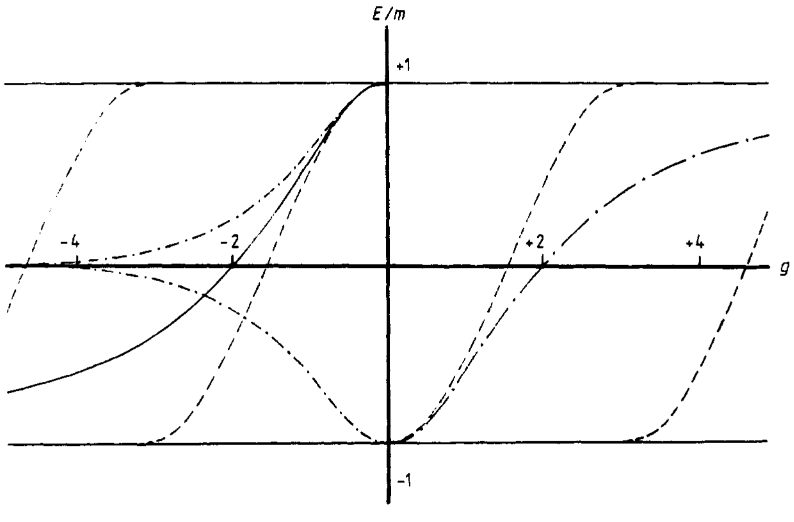

Figure 1. Bound-state energy as a function of the potential strength for pure vector (broken curve), pure scalar (short chain curve), equally mixed (full curve) and inverted mixed (long chain curve) point interaction potentials.

(*d*) Inverted mixed potentials. We have $g_- = \alpha_-$ and $\alpha_+ = 0$. Then we have $q = \frac{1}{2}(m-E)g_-$ with $g_- > 0$ (no bound states appear if $g_- < 0$). The bound-state energy takes the form

$$E = -m(4-g_-^2)/(4+g_-^2) \tag{14}$$

which is positive for $g_- > 2$. This equation is deduced from equation (13) replacing m by $-m$, which is equivalent to changing σ_z and g_s to $-\sigma_z$ and $-g_s$ in the Dirac equation (1). For weak coupling, the bound-state energy is $E \geqslant -m$; by increasing g_- the level is raised but never reaches the upper continuum (figure 1).

The question on confinement by PIPS has been partially answered by McKellar and Stephenson. They pointed out that strong scalar PIPS will confine the particle, whereas the confinement is impossible for strong vector potentials due to the Klein paradox (Klein 1929).

Now we briefly discuss confining properties of equally and inverted mixed potentials. For the sake of completeness, pure vector and pure scalar potentials are also discussed. The probability for particle transmission τ through the potential $U(x)$, defined as $\tau = |T|^2$, can be found to be

$$\tau = \tau(E) = [1+\sin^2(g_v^2-g_s^2)^{1/2}(Eg_s+mg_v)^2/p^2(g_v^2-g_s^2)^{1/2}]^{-1}. \tag{15}$$

For pure vector potentials ($g_s = 0$), τ is bounded from below $\tau \geqslant (1+m^2/p^2)^{-1} > 0$, so particles cannot be confined. Moreover, one can easily check that τ equals unity for the critical values $g_v = n\pi$, so the potential is transparent to all energies. In contrast, for pure scalar potentials ($g_v = 0$), the transmission coefficient becomes $\tau(E) = [1+(E^2/p^2)\sinh^2 g_s]^{-1}$ approaching zero as $|g_s| \to \infty$, leading to particle confinement, i.e. a particle moving in the left (or right) region will remain there indefinitely. For equally and inverted mixed potentials, the transmission coefficient is written as $\tau(E) = [1+(g_\pm/2)^2(E\pm m)/(E\mp m)]^{-1}$, where the upper and the lower signs refer to $g_v = g_s \equiv g_+/2$ and $g_v = -g_s \equiv g_-/2$ respectively. At high energies we have $\tau \sim (1+g^2/4)^{-1}$, and hence the transmission does not occur as the potential becomes sufficiently strong. Therefore, we are brought to the conclusion that both equally and inverted mixed potentials do confine particles. We also note that confining properties depend neither on the sign of the particle energy nor on the character (attractive or repulsive) of the potentials, due to the fact that PIPS of either sign act as a barrier. These conclusions still remain valid in the massless limit, so light particles may be also confined.

The authors thank Dr M T Pérez-Frías and Dr P Fernández for helpful comments.

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